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Note on Determinants of Powers.

By O. H. Mitchell.

If $\alpha > \beta > \gamma > \delta$, then

$$\begin{vmatrix} a^{\alpha} & a^{\beta} & a^{\gamma} & a^{\delta} \\ b^{\alpha} & b^{\beta} & b^{\gamma} & b^{\delta} \\ c^{\alpha} & c^{\beta} & c^{\gamma} & c^{\delta} \\ d^{\alpha} & d^{\beta} & d^{\gamma} & d^{\delta} \end{vmatrix} \div \begin{vmatrix} a^{3} & a^{2} & a^{1} & a^{0} \\ b^{3} & b^{2} & b^{1} & b^{0} \\ c^{3} & c^{2} & c^{1} & c^{0} \\ d^{3} & d^{2} & d^{1} & d^{0} \end{vmatrix} = \begin{vmatrix} \alpha^{3} & \alpha^{2} & \alpha^{1} & \alpha^{0} \\ \beta^{3} & \beta^{2} & \beta^{1} & \beta^{0} \\ \gamma^{3} & \gamma^{2} & \gamma^{1} & \gamma^{0} \\ \delta^{3} & \delta^{2} & \delta^{1} & \delta^{0} \end{vmatrix} \div \begin{vmatrix} 3^{3} & 3^{2} & 3^{1} & 3^{0} \\ 2^{3} & 2^{2} & 2^{1} & 2^{0} \\ 1^{3} & 1^{2} & 1^{1} & 1^{0} \\ 0^{3} & 0^{2} & 0^{1} & 0^{0} \end{vmatrix}$$

terms, *all positive* $(0^0 = 1)$. In briefer notation, representing the first two determinants by their principal diagonals, and writing $\zeta^{\frac{1}{2}}(x, y, z, t)$ for $(x-y)(x-z)(x-t)(y-z)(y-t)(z-t)$,

$$\frac{(a^{\alpha}b^{\beta}c^{\gamma}d^{\delta})}{(a^{3}b^{2}c^{1}d^{0})} = \frac{\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta)}{\zeta^{\frac{1}{2}}(3, 2, 1, 0)} \text{ terms, } \textit{all positive;}$$

and the like is true for determinants of any order. Calling the left-hand member Q_4, the subscript denoting the order of the determinants, Q_4 may be expressed as a linear function with positive coefficients of similar quotients, Q_3; and by successive applications of this formula Q_4 is expressible as $\Sigma a^p b^q c^r d^s$, where p, q, r, s are functions of the two sets of exponents, $\alpha, \beta, \gamma, \delta$ and $3, 2, 1, 0$, and of $(4-1)!$ variables whose limits in the summation are certain functions of $\alpha, \beta, \gamma, \delta$. And the like is true for determinants of any order.

For a proof of the preceding, consider the expression for Q_4 as given by Prof. Cayley, Salmon's Higher Algebra, p. 292, viz:

$$\frac{(a^{\alpha}b^{\beta}c^{\gamma}d^{\delta})}{(a^{3}b^{2}c^{1}d^{0})} = \begin{vmatrix} H_{\alpha-3} & H_{\beta-3} & H_{\gamma-3} & H_{\delta-3} \\ H_{\alpha-2} & H_{\beta-2} & H_{\gamma-2} & H_{\delta-2} \\ H_{\alpha-1} & H_{\beta-1} & H_{\gamma-1} & H_{\delta-1} \\ H_{\alpha} & H_{\beta} & H_{\gamma} & H_{\delta} \end{vmatrix},$$

where $H_m =$ the sum of all possible powers and products of a, b, c, d of the m^{th} degree. If $\delta = 0$, the last row and the last column may be cast off, since $H_0 = 1$, and $H_{-m} = 0$. The left-hand member is divisible by $a^\delta b^\delta c^\delta d^\delta$; thus

$$\frac{(a^\alpha b^\beta c^\gamma d^\delta)}{(a^3 b^2 c^1 d^0)} = a^\delta b^\delta c^\delta d^\delta \left(H_{\alpha-\delta-3}\, H_{\beta-\delta-2}\, H_{\gamma-\delta-1}\right).$$

Expanding each element of the determinant on the right-hand side by means of the formula

$$H_m(a\,,\ b\,,\ c\,,\ d) = d^m + d^{m-1} H_1(a\,,\ b\,,\ c) + d^{m-2} H_2(a\,,\ b\,,\ c) + \text{etc.},$$

and resolving the determinant into the sum of similar determinants, we get

$$\frac{(a^\alpha b^\beta c^\gamma d^\delta)}{(a^3 b^2 c^1 d^0)} = a^\delta b^\delta c^\delta d^\delta \sum_{x=0}^{x=\alpha-\delta-1} \ \sum_{y=0}^{y=\beta-\delta-1} \ \sum_{z=0}^{z=\gamma-\delta-1} d^\theta \begin{vmatrix} H_{x-2} & H_{y-2} & H_{z-2} \\ H_{x-1} & H_{y-1} & H_{z-1} \\ H_x & H_y & H_z \end{vmatrix}_{(a,\ b,\ c)}$$

where the H's now involve only a, b, c, and $\theta = \alpha + \beta + \gamma + \delta - 3 - 4\delta - (x + y + z)$. A little consideration shows that the lower limit of x may be put $= \beta - \delta$ and the lower limit of y put $= \gamma - \delta$ without changing the value of the right-hand member; since the determinants obtained in the summation for lower values of x and y either vanish identically, having two columns the same, or destroy one another by twos, the members of each pair differing from each other only by an interchange of two columns. In fact, whatever x may be, the determinant obtained by putting $y = m$, $z = m$, vanishes identically, while that yielded by putting $y = m$, $z = n$, is destroyed by the one having $y = n$, $z = m$, the variable multiplier, d^θ, being the same for each. Of those determinants which remain, the one obtained by putting $x = l$ and either y or $z = l$ vanishes identically, and the one having $x = l$, $y = m$, $z = n$, is destroyed by that one whose $x = m$, $y = l$, $z = n$, if $\beta - \delta > l > \gamma - \delta - 1$, but by that one of which $x = n$, $y = m$, $z = l$, if $l < \gamma - \delta$; the variable coefficient, d^θ, does not interfere to prevent this mutual destruction of the quantities obtained in the summation, since $\theta = \alpha + \beta + \gamma - 3\delta - 3 - (x + y + z)$. For those determinants which now remain, $x > y > z$, and we have

$$\frac{(a^\alpha b^\beta c^\gamma d^\delta)}{(a^3 b^2 c^1 d^0)} = a^\delta b^\delta c^\delta d^\delta \sum_{x=\beta-\delta}^{x=\alpha-\delta-1} \ \sum_{y=\gamma-\delta}^{y=\beta-\delta-1} \ \sum_{z=0}^{z=\gamma-\delta-1} d^\theta \, \frac{(a^x b^y c^z)}{(a^2 b^1 c^0)}.$$

By successive applications of this formula we get,

$$\frac{(a^{\alpha_1}b^{\alpha_2}c^{\alpha_3}d^{\alpha_4})}{(a^3b^2c^1d^0)}=\sum_{x_1=\alpha_2-\alpha_4}^{x_1=\alpha_1-\alpha_4-1}\ \sum_{x_2=\alpha_3-\alpha_4}^{x_2=\alpha_2-\alpha_4-1}\ \sum_{x_3=0}^{x_3=\alpha_3-\alpha_4-1}\ \sum_{y_1=x_2-x_3}^{y_1=x_1-x_3-1}\ \sum_{y_2=0}^{y_2=x_2-x_3-1}\ \sum_{z_1=0}^{z_1=y_1-y_2-1}a^pb^qc^rd^s,$$

where

$$\begin{aligned}p&=(\alpha_1+\alpha_2+\alpha_3+\alpha_4-3)-4\alpha_4-(x_1+x_2+x_3)+(\alpha_4),\\q&=(x_1+x_2+x_3-2)-3x_3-(y_1+y_2)+(\alpha_4+x_3),\\r&=(y_1+y_2-1)-2y_2-(z_1)+(\alpha_4+x_3+y_2),\\s&=(z_1-0)-z_1-(0)+(\alpha_4+x_3+y_2+z_1).\end{aligned}$$

The notation is slightly changed, and the values of p, q, r, s are written in a somewhat redundant form, in order to show more clearly the law for determinants of higher orders.

Since the number of terms in the H_m of n letters is expressed by the binomial coefficient C_{n-1}^{m+n-1}, the number of terms in the summation above (*i. e.* the sum of the coefficients, all of which are positive) is equal to the determinant

$$(C_3^{\alpha_1}C_3^{\alpha_2+1}C_3^{\alpha_3+2}C_3^{\alpha_4+3}),=(C_3^{\alpha_1}C_2^{\alpha_2}C_1^{\alpha_3}C_0^{\alpha_4}),=\frac{\zeta^{\frac{1}{2}}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}{\zeta^{\frac{1}{2}}(3,2,1,0)},$$

obtained from the H-determinant. As an example,

$$\frac{(a^7b^3c^0)}{(a^2b^1c^0)}=\Sigma a^5b^2+\Sigma a^5bc+\Sigma a^4b^3+2\Sigma a^4b^2c+2\Sigma a^3b^3c+3\Sigma a^3b^2c^2,$$

which has 42, $=\dfrac{\zeta^{\frac{1}{2}}(7,3,0)}{\zeta^{\frac{1}{2}}(2,1,0)}$, terms.

If a_1, a_2, etc., be the n roots of $x^n-P_1x^{n-1}+P_2x^{n-2}-\text{etc.}=0$, by substituting the roots and solving for P_r we get

$$P_r=\frac{(a_1^na_2^{n-1}\ .\ .\ .\ a_r^{n-r+1}a_{r-1}^{n-r-1}\ .\ .\ .\ a_{n-1}^1a_n^0)}{(a_1^{n-1}a_2^{n-2}\ .\ .\ .\ a_r^{n-r}a_{r-1}^{n-r-1}\ .\ .\ .\ a_{n-1}^1a_n^0)}.$$

Thus $\dfrac{(a^4b^3c^2d^0)}{(a^3b^2c^1d^0)}=\Sigma abc$, and since the number of terms in Σabc is C_3^4, we get the following expression for the value of the binomial coefficient, C_3^4, viz:

$$C_3^4=\frac{\zeta^{\frac{1}{2}}(4,3,2,0)}{\zeta^{\frac{1}{2}}(3,2,1,0)}.$$

And, in general, we have

$$C_r^n=\frac{\zeta^{\frac{1}{2}}(n,n-1,\ .\ .\ .\ n-r+1,n-r-1,\ .\ .\ .\ 2,1,0)}{\zeta^{\frac{1}{2}}(n-1,n-2,\ .\ .\ .\ n-r,n-r-1,\ .\ .\ .\ 2,1,0)},$$

the direct proof of which is of course very simple. Having proved that C_r^n has this value, it is easily shown, without reference to the roots of an equation, that P_r, *i. e.* $\Sigma a_1 a_2 \ldots a_r$, has the value obtained above. Thus, taking a special case for brevity, $\frac{(a^4 b^3 c^2 d^0)}{(a^3 b^2 c^1 d^0)}$ is seen to be a symmetric function, of the third degree, in which such a term as abc occurs *at least* once. Σabc is also such a function, in which abc occurs only once. Hence their equality is proved when it is shown that the number of terms in each is the same.

It may be remarked that if the last row of $(a^\alpha b^\beta c^\gamma)$ be subtracted from each of the preceding rows, and these rows be then divided respectively by $a - c$, $b - c$, and the second row be then subtracted from the first and the result divided by $a - b$, we get

$$\frac{(a^\alpha b^\beta c^\gamma)}{(a^2 b^1 c^0)} = \begin{vmatrix} H_{\alpha-2}(a, b, c), & H_{\beta-2}(a, b, c), & H_{\gamma-2}(a, b, c) \\ H_{\alpha-1}(b, c) \quad , & H_{\beta-1}(b, c) \quad , & H_{\gamma-1}(b, c) \\ H_\alpha(c) \quad , & H_\beta(c) \quad , & H_\gamma(c) \end{vmatrix}$$

and since $bH_{m-1}(b, c) + H_m(c) = H_m(b, c)$, and $aH_{m-1}(a, b, c) + H_m(b, c) = H_m(a, b, c)$, etc., the determinant on the right is at once reduced by combination of rows to the similar determinant already considered, in which each row involves every letter, and which is obtained in the text-books, as far as I have observed, only by an indirect process. The method obviously applies to determinants of any order. The following evident identity is used in the reduction:

$$H_m(a, c, d, e, \ldots z) - H_m(b, c, d, e, \ldots z) = (a - b)\, H_{m-1}(a, b, c, \ldots z).$$

The above process of reduction shows that the value of the H-determinant is the same for every distribution of the letters among the rows, subject to the conditions (1) that the letters of every row are contained in each of the preceding rows, and (2) that no row contains *less* than $\omega - r + 1$ letters, where ω is the order of the determinant, and r is the number of the row, counting from the top.

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